GCE Examinations

Advanced Subsidiary / Advanced Level

Further Pure Mathematics
Module FP3

Paper F

MARKING GUIDE

This guide is intended to be as helpful as possible to teachers by providing concise solutions and indicating how marks should be awarded. There are obviously alternative methods that would also gain full marks.

Method marks (M) are awarded for knowing and using a method.

Accuracy marks (A) can only be awarded when a correct method has been used.

(B) marks are independent of method marks.

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FP3 Paper F – Marking Guide

1.	assume true for $n = k$ $\therefore \sum_{r=1}^{k} \ln \frac{r+1}{r} = \ln (k+1)$		
	$\therefore \sum_{r=1}^{k+1} \ln \frac{r+1}{r} = \ln (k+1) + \ln \frac{k+2}{k+1}$	M1 A1	
	$= \ln \frac{(k+1)(k+2)}{k+1} = \ln (k+2)$	M1	
	$= \ln [(k+1)+1]$	A1	
	$\therefore$ true for $n = k + 1$ if true for $n = k$		
	if $n = 1$ $\sum_{r=1}^{n} \ln \frac{r+1}{r} = \ln \frac{2}{1} = \ln 2$, $\ln (n+1) = \ln 2$ $\therefore$ true for $n = 1$	B1	
	$\therefore$ by induction true for $n \in \mathbb{Z}^+$	A1	**(6)**
2.	*(a)* $\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$	M1	
	$\therefore (2-\lambda)(-6-\lambda) - 9 = 0$	A1	
	$\lambda^2 + 4\lambda - 21 = 0$		
	$(\lambda+7)(\lambda-3) = 0$ $\therefore \lambda = -7$ or 3	M1 A1	
	(b) $\lambda = 3$, $\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $-x+3y=0$ $\therefore 3y = x$ $\therefore$ eigenvector $k\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	M1 A1	
	$\lambda = -7$, $\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $3x+y=0$ $\therefore y = -3x$ $\therefore$ eigenvector $k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$	M1 A1	**(8)**
3.	*(a)* $w(z - \mathrm{i}) = z + 2\mathrm{i}$; $wz - w\mathrm{i} = z + 2\mathrm{i}$	M1	
	$z(w-1) = w\mathrm{i} + 2\mathrm{i}$ $\therefore z = \frac{\mathrm{i}(w+2)}{w-1}$	A1	
	$\lvert z \rvert = 1$ $\therefore \lvert \mathrm{i} \rvert \lvert w+2 \rvert = \lvert w-1 \rvert$	M1	
	$\lvert w+2 \rvert = \lvert w-1 \rvert$	A1	
	$\therefore$ perp. bisector of $-2 + 0\mathrm{i}$ and $1 + 0\mathrm{i}$ $\therefore u = -\frac{1}{2}$	A1	
	(b) $\lvert w \rvert = 2$, $\left\lvert \frac{z+2\mathrm{i}}{z-\mathrm{i}} \right\rvert = 2$		
	$\therefore \lvert z+2\mathrm{i} \rvert = 2\lvert z-\mathrm{i} \rvert$	M1	
	$x^2 + (y+2)^2 = 4x^2 + 4(y-1)^2$	A1	
	$x^2 + y^2 + 4y + 4 = 4x^2 + 4y^2 - 8y + 4$		
	$3x^2 + 3y^2 - 12y = 0$		
	$x^2 + y^2 - 4y = 0$	A1	
	$x^2 + (y-2)^2 - 4 = 0$ or $x^2 + (y-2)^2 = 4$	M1	
	$\therefore$ circle, centre $0 + 2\mathrm{i}$, radius 2	A1	
	$a = 0$, $b = 2$, $r = 2$	A1	**(11)**

4. (a)	$y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_0 + \frac{1}{2}(x - x_0)^2\left(\frac{d^2y}{dx^2}\right)_0 + \ldots$	B1	
	$x = x_0 + h,\ y_1 \approx y_0 + h\left(\frac{dy}{dx}\right)_0 + \frac{1}{2}h^2\left(\frac{d^2y}{dx^2}\right)_0$ (I)		
	$x = x_0 - h,\ y_{-1} \approx y_0 - h\left(\frac{dy}{dx}\right)_0 + \frac{1}{2}h^2\left(\frac{d^2y}{dx^2}\right)_0$ (II)	M1 A1	
	(I) + (II) $y_1 + y_{-1} \approx 2y_0 + h^2\left(\frac{d^2y}{dx^2}\right)_0$ giving $\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$	M1 A1	
(b)	$\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} - 3y = 0$		
	$\frac{y_1 - 2y_0 + y_{-1}}{0.01} + (x_0 + 2)\frac{y_1 - y_{-1}}{0.2} - 3y_0 = 0$	M1 A1	
	$x_{-1} = 0, x_0 = 0.1, x_1 = 0.2;\ y_{-1} = 1, y_0 = 1.2, y_1 = ?$		
	$100(y_1 - 2.4 + 1) + 5(0.1 + 2)(y_1 - 1) - 3.6 = 0$	M1 A1	
	giving $110.5y_1 = 154.1 \ \therefore\ y_1 = 1.39457\ldots = 1.39$ (3sf)	M1 A1	**(11)**

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5. (a)	$\det \mathbf{A} = 1(-q-2) + 1(-4-1) + 3(8-q) = 17 - 4q$	M1 A1	
	matrix of cofactors: $\begin{pmatrix} -q-2 & 5 & 8-q \\ 5 & -4 & -3 \\ -1-3q & 11 & q+4 \end{pmatrix}$	M1 A2	
	$\therefore\ \mathbf{A}^{-1} = \frac{1}{17-4q}\begin{pmatrix} -q-2 & 5 & -1-3q \\ 5 & -4 & 11 \\ 8-q & -3 & q+4 \end{pmatrix}$	M1 A1	
(b)	$\begin{pmatrix} 1 & -1 & 3 \\ 4 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ $q = 1$	M1	
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -3 & 5 & -4 \\ 5 & -4 & 11 \\ 7 & -3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -13 \\ 52 \\ 26 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$	M1 A1	
	$x = -1, y = 4, z = 2$	A1	**(11)**

6.	*(a)*	$\frac{dy}{dx} = \sqrt{1-x^2} \times \frac{-1}{\sqrt{1-x^2}} + \arccos x \times \frac{1}{2} \frac{1}{\sqrt{1-x^2}}(-2x)$	M1 A1	
		$\frac{dy}{dx} = -1 - \frac{y}{\sqrt{1-x^2}} \times \frac{x}{\sqrt{1-x^2}}$	M1	
		$(1-x^2)\frac{dy}{dx} = -(1-x^2) - xy$	M1	
		$(1-x^2)\frac{dy}{dx} + xy - x^2 + 1 = 0$	A1	
	(b)	$(1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) + x\frac{dy}{dx} + y - 2x = 0$	M1 A1	
		$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y - 2x = 0$		
		$(1-x^2)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}(-2x) - x\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{dy}{dx} - 2 = 0$	M1 A1	
		$(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} - 2 = 0$		
		$y_0 = 1 \times \arccos 0 = \frac{\pi}{2}$; $1\left(\frac{dy}{dx}\right)_0 + 0 - 0 + 1 = 0 \;\therefore \left(\frac{dy}{dx}\right)_0 = -1$	A1	
		$1\left(\frac{d^2y}{dx^2}\right)_0 - 0 + \frac{\pi}{2} - 0 = 0 \;\therefore \left(\frac{d^2y}{dx^2}\right)_0 = -\frac{\pi}{2}$		
		$1\left(\frac{d^3y}{dx^3}\right)_0 - 0 - 2 = 0 \;\therefore \left(\frac{d^3y}{dx^3}\right)_0 = 2$	A1	
		$\therefore y = \frac{\pi}{2} - x - \frac{\pi}{4}x^2 + \frac{1}{3}x^3 + \ldots$	M1 A1	**(13)**

7.	*(a)*	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$		
		$\mathbf{n} = \mathbf{i}(1-2) - \mathbf{j}(0-2) + \mathbf{k}(0-1) = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$	M1 A2	
	(b)	$\mathbf{r}.(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (3\mathbf{i} + \mathbf{j} - 4\mathbf{k}).(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -3 + 2 + 4 = 3$ $\mathbf{r}.(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3$	M1 A1	
	(c)	$\Pi_1 : \mathbf{r}.\frac{-\mathbf{i}+2\mathbf{j}-\mathbf{k}}{\sqrt{6}} = \frac{3}{\sqrt{6}}$	B1	
		plane parallel to Π_1 through A:		
		$\mathbf{r}.(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}).(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -2 + 2 - 4 = -4$	M1	
		$\therefore \mathbf{r}.\frac{-\mathbf{i}+2\mathbf{j}-\mathbf{k}}{\sqrt{6}} = \frac{-4}{\sqrt{6}}$	A1	
		$\therefore$ distance A to $\Pi_1 = \frac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6}$	A1	
	(d)	$\lvert(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}).(\mathbf{i} + b\mathbf{j})\rvert = \sqrt{6}\sqrt{(1 + b^2)}\cos 30°$	M1 A1	
		$\lvert -1 + 2b \rvert = \sqrt{6}\sqrt{(1 + b^2)}\frac{\sqrt{3}}{2}$	A1	
		$(2b-1)^2 = \frac{18}{4}(1 + b^2)$	M1	
		$2(4b^2 - 4b + 1) = 9(1 + b^2)$ giving $b^2 + 8b + 7 = 0$		
		$(b+1)(b+7) = 0 \;\therefore b = -1$ or -7	M1 A1	**(15)**

	Total	**(75)**

Performance Record – FP3 Paper F

Question no.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Total
Topic(s)	proof by induction	matrices, eigenvals.	complex trans.	diff. eqn., Taylor series, step-by-step	matrices, inverse	Maclaurin series	vectors	
Marks	6	8	11	11	11	13	15	75
Student								